Balanced Proper Orthogonal Decomposition applied to magnetoquasistatic problems

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Model Order Reduction (MOR) methods are an active research field in the numerical analysis domain. They are applied to many different areas in physics, especially in mechanics because they allow to dramatically reduce the computational time. MOR is quite recent in electromagnetics and needs still to be investigated. The Proper Orthogonal Decomposition (POD) is the most famous one and has already shown very promising results. However, the POD approach minimizes the error in the L^2 sense on the whole domain and cannot be very accurate to calculate quantities of interest, like flux associated with a probe in region where the field is low. In this communication, we present the Balanced Proper Orthogonal Decomposition (BPOD) which extends the POD by taking account of probes in its model. The BPOD and POD approaches will be compared on a 3D linear magnetoquasistatic field problem.

***Index Terms*—Balanced Proper Orthogonal Decompoition, Balanced Truncation, Model Order Reduction, Proper Orthogonal Decomposition**

I. Introduction

APPLYING the Finite Element Method (FEM) coupled with a time-stepping scheme is increasingly used to model electromagnetic devices. This approach enables to obtain very accurate results but requires to solve large scale systems, leading to a significant computational cost. Many model order reduction (MOR) methods have been developed over the past few years in order to overcome this problem. Most of them consist in looking for the solution in a reduced basis which highly reduces the size of the full problem. Therefore, the key of MOR methods is to build the most suitable reduced basis for a given problem. The Proper Orthogonal Decomposition (POD) method is the most popular MOR approach. POD is very efficient in reducing computation time of linear and nonlinear problems [1]. However, the POD method consists in minimizing the error on the whole domain. Thus, it leads to a lack of accuracy when calculating local quantities of interest, such as magnetic flux in probes. In this context, the Balanced Proper Orthogonal Decomposition (BPOD) can be applied. This approach which has been developed in mechanics and fluid mechanics, but not yet in electromagnetics, enables to build a reduced basis according to the quantity of interest (the probe signal), increasing the accuracy of this quantity [2].

In this communication, we propose to develop and to compare the BPOD and POD approaches in the case of a magnetoquasistatic problem when the quantity of interest is a probe signal. First, the numerical model obtained from the modified vector potential formulation is briefly presented. Then, the BPOD and POD methods are developed. Finally, an academic example is studied with both methods, and compared in terms of accuracy with the full Finite Element Model.

II. Model Order Reduction with BPOD and POD

Let us consider a magnetoquasistatic problem in a domain D. A conducting domain D_c is included in D. For sake of clarity, we assume that the domain D contains two stranded inductors, even though the following approach remains valid with more inductors. The first one is supplied by a current $i(t)$ and the second one is not supplied but is used as a flux probe. The flux, the quantity of interest, is denoted $\Phi(t)$. This problem can be solved with the Finite Element Method by using the modified magnetic vector potential formulation. Thus, we obtain a system of algebro-differential equations:

$$\boldsymbol{N}\frac{\mathrm{d}\boldsymbol{X}(t)}{\mathrm{d}t} + \boldsymbol{M}\boldsymbol{X}(t) = \boldsymbol{F}_{src}\ i(t) \tag{1}$$

$$\Phi(t) = \boldsymbol{F}_{prb}^{t}\boldsymbol{X}(t) \tag{2}$$

with $\boldsymbol{X}(t)$ the vector solution of size n, $\boldsymbol{M}$ and $\boldsymbol{N}$ square matrices depending on the magnetic permeability and the electric conductivity respectively, $\boldsymbol{F}_{src}$ and $\boldsymbol{F}_{prb}$ vectors accounting for the source and the probe inductors.

A. Balanced Proper Orthogonal Decomposition

The BPOD approach derives from Balanced Truncation introduced by Moore [3] which consists in reducing a numerical model by considering the controllability and the observability of the system. In order to obtain an approximation of the controllability $\mathcal{G}_{cont}$ and the observability $\mathcal{G}_{obs}$ Grammians, the snapshots method is used [4] [5].

1) Snapshots method

To compute an approximation of $\mathcal{G}_{cont}$ and $\mathcal{G}_{obs}$, the Fourier transform of the problem (1) is used and the snapshot method [6] is applied. First, the primal system (1) is solved in the frequency domain for m different frequencies:

$$\boldsymbol{X}_{src}^{k} = (j\omega_k\boldsymbol{N} + \boldsymbol{M})^{-1}\,\boldsymbol{F}_{src} \tag{3}$$

Let us define the source snapshots matrix $\boldsymbol{X}_{src} \in \mathbb{R}^{n\times 2m}$ by concatenating the real and imaginary part of the m column-vectors $\boldsymbol{X}_{src}^{k},\ k = 1\ldots m$. The controllability Grammian is then approximated by $\mathcal{G}_{cont} \approx \boldsymbol{X}_{src}\ \boldsymbol{X}_{src}^{t}$.

The observability Grammian $\mathcal{G}_{obs} \in \mathbb{R}^{n\times n}$ is approximated by applying the same method to the dual system of (1–2):

$$\boldsymbol{N}^t \frac{\mathrm{d}\boldsymbol{X}(t)}{\mathrm{d}t} + \boldsymbol{M}^t \boldsymbol{X}(t) = \boldsymbol{F}_{prb}\; i(t) \tag{4}$$

$$\Phi_{dual}(t) = \boldsymbol{F}^t_{src} \boldsymbol{X}(t) \tag{5}$$

Equation (4) is also solved in the frequency domain:

$$\boldsymbol{X}^k_{prb} = \left(j\omega_k \boldsymbol{N}^t + \boldsymbol{M}^t\right)^{-1} \boldsymbol{F}_{prb} \tag{6}$$

Then, the probe snapshots matrix $\boldsymbol{X}_{prb} \in \mathbb{R}^{n\times 2m}$ is defined by concatenating the real and imaginary part of $\boldsymbol{X}^k_{prb},\ k = 1\ldots m$, and $\mathcal{G}_{obs} \approx \boldsymbol{X}_{prb}\ \boldsymbol{X}^t_{prb}$.

2) Reduced basis

Performing a Singular Value Decomposition (SVD) on the low rank matrix $(\boldsymbol{X}^t_{prb}\ \boldsymbol{X}_{src}) \in \mathbb{R}^{2m\times 2m}$ allows to generate a reduced basis. Therefore, $\boldsymbol{X}^t_{prb}\ \boldsymbol{X}_{src} = \boldsymbol{U\Sigma V}^t$ with $\boldsymbol{U}$ and $\boldsymbol{V}$ unitary matrices in $\mathbb{R}^{2m\times 2m}$, and $\boldsymbol{\Sigma}$ a diagonal matrix in $\mathbb{R}^{2m\times 2m}$. Finally, balanced controllable and observable modes, $\boldsymbol{T} \in \mathbb{R}^{n\times 2m}$ and $\boldsymbol{S} \in \mathbb{R}^{2m\times n}$, are defined by

$$\boldsymbol{T} = \boldsymbol{X}_{src}\boldsymbol{V}\boldsymbol{\Sigma}^{-1/2} \quad \text{and} \quad \boldsymbol{S} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{U}^t\boldsymbol{X}^t_{prb} \tag{7}$$

3) Reduced model with BPOD

Using a Petrov-Galerkin procedure on the system (1–2) leads to the following reduced model of size $2m$:

$$\boldsymbol{N}_r \frac{\mathrm{d}\boldsymbol{X}_r(t)}{\mathrm{d}t} + \boldsymbol{M}_r \boldsymbol{X}_r(t) = \boldsymbol{F}_{src,r}\; i(t) \tag{8}$$

$$\Phi(t) = \boldsymbol{F}^t_{prb,r}\ \boldsymbol{X}_r(t) \tag{9}$$

where $\boldsymbol{N}_r = \boldsymbol{SNT}$, $\boldsymbol{M}_r = \boldsymbol{SMT}$ are $2m \times 2m$ matrices. $\boldsymbol{F}_{src,r} = \boldsymbol{SF}_{src}$, $\boldsymbol{F}_{prb,r} = \boldsymbol{T}^t\boldsymbol{F}_{prb}$ and $\boldsymbol{X}_r$ are vectors of size $2m$.

B. Proper Orthogonal Decomposition

The previous framework allows to generate a reduced system through the classical POD technique quite easily. Indeed, the POD approach does not consider the probe, and neither the dual system (4–5). Therefore, performing a SVD on $(\boldsymbol{X}^t_{src}\ \boldsymbol{X}_{src})$ such that $(\boldsymbol{X}^t_{src}\ \boldsymbol{X}_{src}) = \boldsymbol{U\Sigma V}^t$ allows to build the reduced problem (8–9) with $\boldsymbol{S} = \boldsymbol{T}^t$.

III. Application

A 3D linear magnetodynamic problem composed of an aluminium conducting plate, a probe inductor and a source inductor supplied by a square wave current with a frequency $f_0 = 1$kHz is studied. The mesh is made of 12593 nodes and 68835 tetrahedrons. The backward Euler method is applied on (1) in order to solve the problem on six periods with a 25μs time step. Reduced basis obtained with POD and BPOD are computed in the frequency domain at the five training angular velocities: $\omega_k = 2k\pi f_0,\ k = 1\ldots 5$. The solutions of (8–9) are computed in the time domain using BPOD and POD methods, and then compared to a full Finite Element model. Figure 2 presents the relative error $\mathcal{E}(\Phi) = \frac{\|\Phi_{ref}(t)-\Phi_{red}(t)\|}{\|\Phi_{ref}(t)\|}$ on the magnetic flux associated with the probe inductor obtained from both MOR methods, versus the size of the reduced basis. Figure 3 shows the evolution of the magnetic flux computed in the probe by the BPOD and the POD models, for a reduced basis of size 8. With this size of reduced basis, the speedup factors are 36 and 21 for POD and BPOD respectively. Speedup is calculated by taking account of the snapshots computational cost. It appears that the POD is the fastest method whereas the BPOD model offers a good compromise between modeling the quantity of interest with accuracy and a significant speedup.

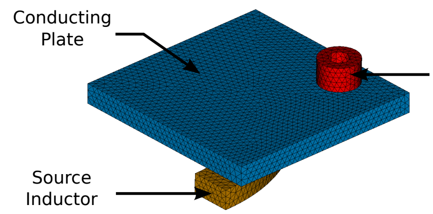

Fig. 1. 3D mesh of the problem

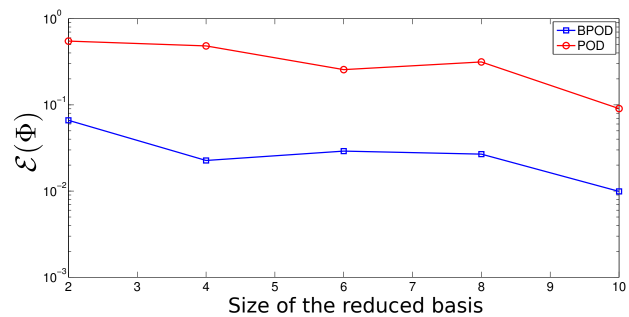

Fig. 2. Relative error of the magnetic flux associated with the probe versus the size of the reduced basis (log scale)

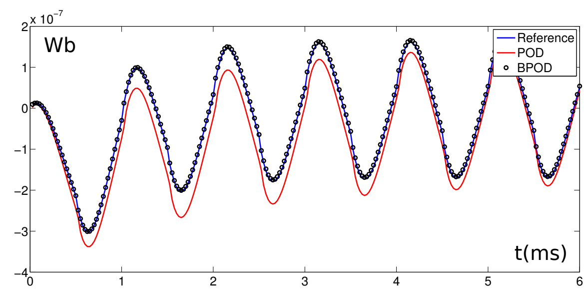

Fig. 3. Magnetic flux associated with the probe inductor

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